Section 1.4: Continuity and One-Sided Limits

Continuity

A function is said to be continuous over an open interval (*a*, *b*) if for every point *c* on that interval, $\lim_{x \to c} f(x) = f(c)$.

In other words, the value the function is approaching is equal to the value at the destination; there are no sudden jumps or holes. A continuous function can be written without lifting one's pencil from the paper.

A function is said to be continuous over the *closed* interval [*a*, *b*] if it is continuous over the open interval (*a,* b) and additionally $\lim_{x \to a+} f(x) = f(a)$ and $\lim_{x \to b-} f(x) = f(b)$.

In particular, all polynomial functions $f(x) = a x^n + b x^{n-1} + c x^{n-2}$... are continuous everywhere. Moreover, radical functions of the form $f(x) = \sqrt[n]{x}$ are continuous everywhere if *n* is negative, and continuous for x ≥ 0 if *n* is otherwise.

If *f*(*x*) and *g*(*x*) are continuous functions over a certain interval, then the composite function $f \circ g = f(g(x))$ is also continuous over that same interval.

Discontinuities

If $\lim_{x \to c} f(x) \neq f(c)$, *f*(*x*) is said to have a *discontinuity* at *c.* There are several different types of discontinuities.

Removable discontinuity: a function is continuous everywhere except at one point at which the function is not defined or does not equal its limit. Example:

$$f(x) = \frac{x^2 - 1}{x - 1} \quad \text{at } x = 1$$

Jump discontinuity: the left hand limit and right hand limit exist but do not equal each other. Example:

$$f(x) = \begin{cases} 3, & x \leq 1 \\ -5, & x > 1 \end{cases} \quad \text{at } x = 1$$

Infinite discontinuity: the function has a vertical asymptote (see next section). Example:

$$f(x) = \frac{1}{x} \text{ or } \frac{1}{x^2} \quad \text{at } x = 1$$

Oscillating discontinuity: as x approaches a particular number the frequency of a sinusoidal function approaches infinity. Example:

$$\sin \frac{1}{x} \quad \text{at } x = 0$$

Intermediate value theorem

The intermediate value theorem (IVM) states that "if *f*(*x*) is continuous over the closed interval [*a*, *b*] and *k* is any number between *f*(*a*) and *f*(*b*), then there must be some *x*-value *c* such that *f*(*c*) = *k.*"

The IVM can be paraphrased roughly as follows: if *k* is on a path between *f*(*a*) and *f*(*b*), then any transit from *f*(*a*) to *f*(*b*) must pass through k.

The IVM is often used to prove the existence of roots, because if a function is continuous and *f*(*a*) < 0 and *f*(b) > 0, then there must be some point *c* in between a and *b* such that *f*(*c*) = 0.

Section 1.5: Infinite Limits

If, as *x* approaches a particular point *c,* the corresponding *f*(*x*) becomes an arbitrarily large positive number, then *f*(*x*) is said to have an infinite limit at *x* = *c.* This is written mathematically as

$$\lim_{x \to c} f(x) = \infty$$

If *f*(*x*) instead approaches an arbitrarily large negative number, then

$$\lim_{x \to c} f(x) = -\infty$$

Note that labeling the limit as ∞ or -∞ does not imply that the limit *exists.* Instead, an infinite limit *fails* to exist, because it fails to converge on a specific value, as stated in the definition of a limit as given in section 1.2.

A function that has an infinite limit at a particular point will also have a vertical asymptote. In particular, if *g*(*c*) = 0 and *f*(*c*) ≠ 0, then the function $h(x) = \frac{f(x)}{g(x)}$ has a vertical asymptote at *x* = *c.*